Random diffusion and anomalous roughening

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Presentation of the model

Linear growth equation:

$$\partial_t h(x,t) = \partial_x D(x) \partial_x h(x,t) + \eta(x,t) \tag{1}$$

- $D(x)$ random diffusion coefficient (disorder)
- $\eta(x,t)$ Gaussian white noise (thermal noise).

Motivations:

- Similarities with the harder Porous Medium Equation (PME)
- This interface can present 2 roughness exponents

Discretized version

$$\begin{aligned}\partial_t h_i(t) &= (D_{i+1} - D_i)(h_{i+1} - h_i) + D_i(h_{i+1} + h_{i-1} - 2h_i) + \eta_i(t) \\ &= D_{i+1}h_{i+1} + D_i h_{i-1} - (D_i + D_{i+1})h_i + \eta_i(t) \\ &= \sum_j \Lambda_{ij} h_j(t) + \eta_i(t),\end{aligned} \tag{2}$$

$$P(D_i) = (1-\Phi)D_i^{-\Phi}, \quad 0 < D_i < 1. \tag{3}$$

Two distinct regimes:

- $\Phi < 0$: $D_i \sim 1$ and the interface is smooth
- $\Phi > 0$: we can have $D_i \ll 1$, which leads to "jumps" in the interface

Standard morphology ($\Phi < 0$)

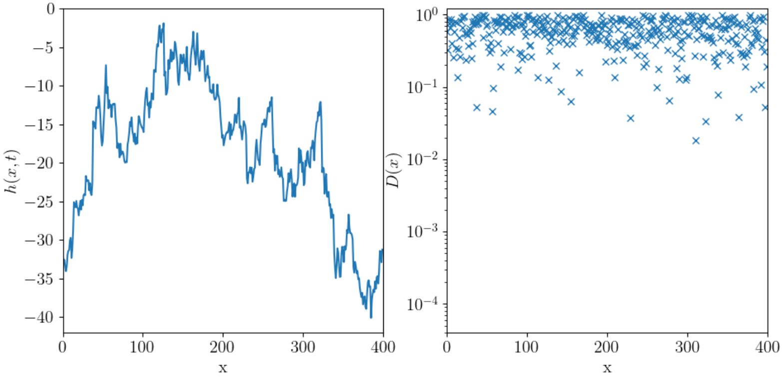


Figure: $\Phi = -0.5$, unique roughness exponent $\zeta = \frac{1}{2}$

Anomalous morphology ($\Phi > 0$)

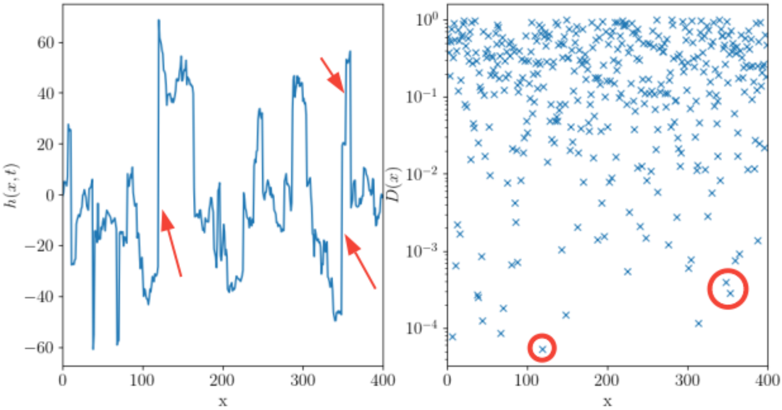
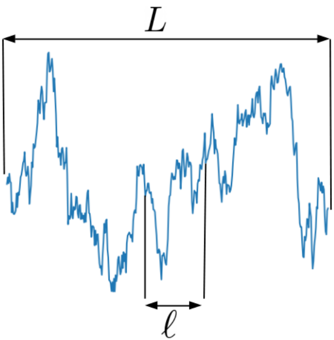


Figure: $\Phi = 0.5$, Large jumps. Two roughness exponents?

The width of the interface



$$w_2(\ell) = \frac{1}{\ell} \sum_{u=x}^{x+\ell} \left[h(u) - \frac{1}{\ell} \sum_{u=x}^{x+\ell} h(u) \right]^2 \tag{4}$$

Two averages:

- $\langle w_2(\ell) \rangle$ thermal average
- $\overline{\langle w_2(\ell) \rangle}$ thermal and disorder average

State of the art [1] (Stationnary state)

$$\overline{\langle w_2(\ell)\rangle} = \begin{cases} \ell^{2\zeta}, \quad \zeta = 1/2 & \text{for} \quad \Phi < 0 \\ \ell^{2\zeta_{\mathrm{loc}}} L^{2(\zeta_{\mathrm{G}}-\zeta_{\mathrm{loc}})}, \quad \zeta_{\mathrm{G}} = 1/(2-2\Phi), \zeta_{\mathrm{loc}} = 1/2 & \text{for} \quad \Phi > 0 \end{cases} \quad (5$$

[1] J. López and M. Rodríguez, Phys. Rev. E **54** R2189, *Lack of self-affinity and anomalous roughening in growth processes*

Explicit solution (Stationnary state)

$$\langle h_i h_j \rangle = \Lambda_{ij}^{-1} \tag{6}$$

$$\langle w_2(\ell) \rangle = \frac{1}{\ell} \sum_{i=1}^{\ell} \langle h_i^2 \rangle - \frac{1}{\ell^2} \sum_{i=1}^{\ell} \sum_{j=1}^{\ell} \langle h_i h_j \rangle . \tag{7}$$

Steady-state solution

Upon solving a recurrence relation, one finds:

$$\Lambda_{ij}^{-1} = \frac{\left(\sum_{k=1}^{i} X_k\right)\left(\sum_{l=j+1}^{L+1} X_l\right)}{\sum_{p=1}^{L+1} X_p} \tag{8}$$

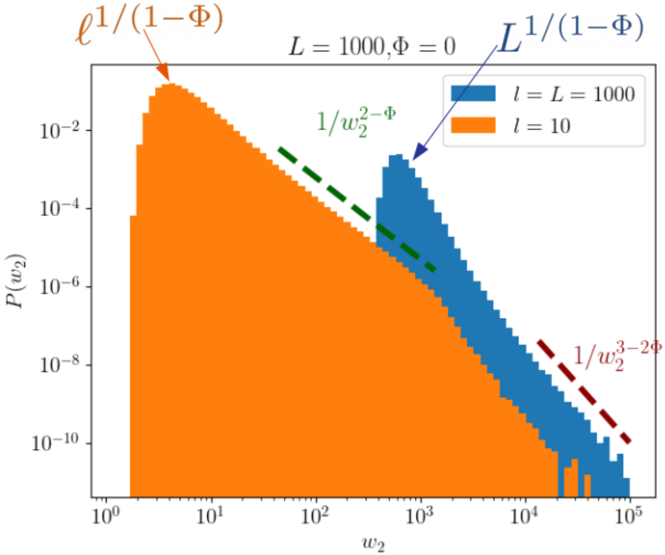
$$X_i = 1/D_i \tag{9}$$

$$P(X_i) = (1-\Phi)/X_i^{2-\Phi} \quad \text{for} \quad X_i > 1 \tag{10}$$

Explicit solution

$$\langle w_2(\ell)\rangle = \frac{(\sum_{j=\ell+1}^{L} X_j)(\sum_{k=0}^{\ell-1}(1-\frac{k}{\ell})\frac{k}{\ell}X_{k+1}) + \sum_{1\leq i<j}^{\ell}(1-\frac{j-i}{\ell})\frac{j-i}{\ell}X_iX_j}{\sum_{k=1}^{L+1} X_k} \tag{11}$$

Distribution of the steady-state solution ($\Phi \geq 0$)



Demonstration of the first asymptotic ($\Phi > 0$)

For $\ell \ll L$, one has:

$$\langle w_2(\ell)\rangle = \frac{(\sum_{j=\ell+1}^{L} X_j)(\sum_{k=0}^{\ell-1}(1-\frac{k}{\ell})\frac{k}{\ell}X_{k+1}) + \sum_{1\leq i<j}^{\ell}(1-\frac{j-i}{\ell})\frac{j-i}{\ell}X_i X_j}{\sum_{p=1}^{\ell} X_p + \sum_{k=\ell+1}^{L+1} X_k} \tag{12}$$

For $\langle w_2\rangle \ll L^{1/(1-\Phi)}$:

$$\langle w_2(\ell)\rangle = \sum_{k=0}^{\ell-1}(1-\frac{k}{\ell})\frac{k}{\ell}X_{k+1} \tag{13}$$

$$P(\langle w_2(\ell)\rangle = w) \sim 1/w^{2-\Phi}. \tag{14}$$

For $\Phi > 0$, the typical values are $\sim \ell^{1/(1-\Phi)}$.

Demonstration of the second asymptotic ($\Phi > 0$)

$$\sum_{k=0}^{\ell-1}(1-\frac{k}{\ell})\frac{k}{\ell}X_{k+1} \simeq \sum_{p=1}^{\ell} X_p \tag{15}$$

For $\langle w_2(\ell)\rangle \gg L^{1/(1-\Phi)}$, we need:

$$\langle w_2(\ell)\rangle = \frac{\overbrace{(\sum_{j=\ell+1}^{L} X_j)}^{L^{1/(1-\Phi)} \ll}\overbrace{(\sum_{p=1}^{\ell} X_p)}^{L^{1/(1-\Phi)} \ll}}{\sum_{p=1}^{\ell} X_p + \sum_{j=\ell+1}^{L+1} X_j}$$

$$\begin{aligned} P(\langle w_2(\ell)\rangle = w) &= N\int_{v=w}^{+\infty} dv \frac{1}{w^{2-\Phi}} \frac{(v-w)^{2-\Phi}}{v^{2(2-\Phi)}} \\ &= N\frac{1}{w^{3-2\Phi}}\int_0^{+\infty} dv \frac{v^{2-\Phi}}{(v+1)^{2(2-\Phi)}}. \end{aligned} \tag{16}$$

Consequences

$\Phi > 0$:

$$P(\langle w_2(\ell)\rangle) \sim \begin{cases} 1/\langle w_2\rangle^{2-\Phi} & \text{for } \ell^{1/(1-\Phi)} \ll \langle w_2\rangle \ll \text{Ł}^{1/(1-\Phi)} \\ 1/\langle w_2\rangle^{3-2\Phi} & \text{for } L^{1/(1-\Phi)} \ll \langle w_2\rangle \end{cases} \quad (17)$$

(Similar results for $\Phi < 0$, with the lower bound in ℓ and the transition at L)

- $\Phi < 0$: $\overline{\langle w_2(\ell)\rangle} \sim \ell^1$
- $0 < \Phi < 0.5$: $\overline{\langle w_2(\ell)\rangle} \sim \ell^1 L^{\Phi/(1-\Phi)}$
- $0.5 < \Phi < 1$: $\overline{\langle w_2(\ell)\rangle}$ not defined

Calculation of the moments

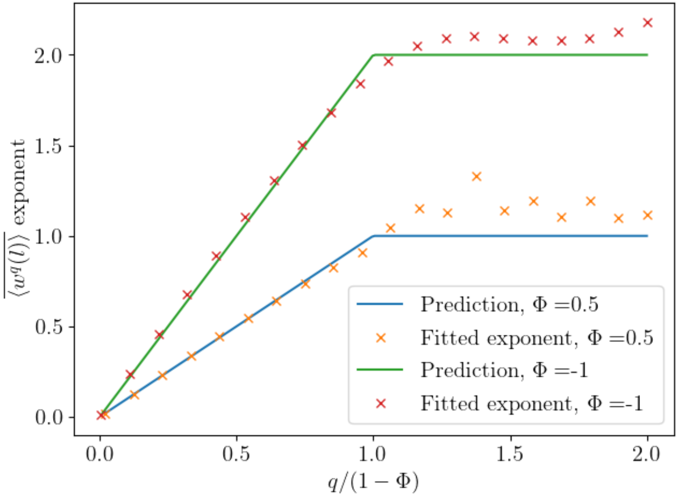
For $q < 1 - \Phi$:

$$\overline{\langle w_2(\ell)\rangle^q} \sim \begin{cases} \ell^q & \text{for } \; \Phi < 0 \\ \\ \ell^{q/(1-\Phi)} & \text{for } \; 0 < \Phi < 1 \end{cases} \tag{18}$$

For $1 - \Phi < q < 2(1 - \Phi)$:

$$\overline{\langle w_2(\ell)\rangle^q} \sim \begin{cases} \ell^{1-\Phi} L^{q-(1-\Phi)} & \text{for } \; \Phi < 0 \\ \\ \ell^1 L^{q/(1-\Phi)-1} & \text{for } \; 0 < \Phi < 1 \end{cases} \tag{19}$$

Numerical estimation of the exponent



Conclusion

Small diffusion coefficient $P(D_i) = (1 - \Phi) D_i^{-\Phi}$ leads to:

- Fat-tailed distribution
- Anomalous scaling ($\Phi > 0$)
- Two roughness exponents (local and global) can be misleading